TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 156, May 1971

SOME IMMERSION THEOREMS FOR MANIFOLDS

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Abstract. In this paper we obtain several results on immersing manifolds into Euclidean spaces. For example, a spin manifold M^n immerses in R^{2n-3} for dimension $n \equiv 0 \bmod 4$ and n not a power of 2. A spin manifold M^n immerses in R^{2n-4} for $n \equiv 7 \bmod 8$ and $n > 7$. Let M^n be a 2-connected manifold for $n \equiv 6 \bmod 8$ and $n > 6$ such that $H_3(M; Z)$ has no 2-torsion. Then M immerses in R^{2n-5} and embeds in R^{2n-4}. The method of proof consists of expressing k-invariants in Postnikov resolutions for the stable normal bundle of a manifold by means of higher order cohomology operations. Properties of the normal bundle are used to evaluate the operations.

1. **Preliminaries.** By a manifold M^n we mean that M is a closed connected smooth manifold of dimension n. We write $M \subseteq R^s$ and $M \subset R^t$ to denote the existence of a differentiable immersion of M into Euclidean s-space and a smooth embedding of M into Euclidean t-space respectively. A manifold M is called a spin manifold iff $w_1(M) = w_2(M) = 0$. The geometric dimension of a stable vector bundle ξ over a complex X, denoted g.dim ξ, is the smallest integer k for which there is a k-plane bundle over X stably isomorphic to ξ. The coefficient group for singular cohomology is understood to be Z_2 whenever omitted. We denote the mod 2 Steenrod algebra by A. $A(Y)$ denotes the semitensor algebra $H^*(Y) \otimes A$ defined in [19] for any space Y. Finally $\alpha(n)$ represents the number of 1's appearing in the dyadic expansion of the positive integer n. In [10] Glover proves that a k-connected manifold M^n embeds in R^{2n-2k} if it immerses in $R^{2n-2k-1}$ for $0 \leqq k \leqq (n-3)/4$. All spaces are assumed to be complexes (pathwise connected CW-complexes with basepoint) and all maps preserve basepoint. The author wishes to express his sincere gratitude to his advisor, Professor Emery Thomas.

A formulation of [18, Theorem II] for spin manifolds is the following

Proposition 1.1. *Let M^n be a spin manifold such that $\bar{w}_{n-k}(M) \neq 0$. There are nonnegative integers a_j for $j \geqq 0$ satisfying the conditions*:

1. $\sum a_j = k$,
2. $\sum 2^j a_j = n$,
3. a_1 *is even*,
4. *if* $a_0 = 0$, *the first nonzero* a_j *and its immediate successor* a_{j+1} *must be even*,
5. *if* a_2 *is even*, $a_1 \equiv 0 \bmod 4$; *if* a_2 *is odd*, $a_1 \equiv 2 \bmod 4$.

Received by the editors May 8, 1969.

AMS 1970 *subject classifications*. Primary 57D40, 55G45, 55G20, 55G40.

Key words and phrases. Immersion, embedding, spin manifold, normal bundle, Postnikov resolution, k-invariant, twisted cohomology operation, higher order cohomology operation, Thom complex, Poincaré duality, Dold manifold.

Proof. Massey and Peterson show in [18] that there exists an admissible monomial Sq^I in A of degree $n-k$ such that $Sq^I x \neq 0$ for some class x in $H^k(M)$. They write $Sq^I = Sq^{i_1} \cdots Sq^{i_p}$ and set $a_j = i_j - 2i_{j+1}$ for $0<j<p$ and $a_p = i_p$. Since M is a spin manifold, the Wu classes $V_s(M)=0$ for $0<s<4$. So by the Adem relations $Sq^t\colon H^{n-t}(M) \to H^n(M)$ is trivial for t not divisible by 4. Thus $i_1 \equiv 0 \bmod 4$ and condition 5 follows. Conditions 1–4 are established in [18].

COROLLARY 1.2. *Let M^n be a spin manifold with $n \equiv j \bmod 8$ for $n>j$ and $3<j<8$. Then $\bar{w}_{n-j}(M)=0$.*

THEOREM 1.3. *Let M^n be a 3-connected manifold with $n \equiv j \bmod 8$, $n>j$, and $3<j<8$. Then $M \subset R^{2n-4}$ and $M \subseteq R^{2n-5}$.*

Proof. Now $\bar{w}_{n-j}(M)=0$ by (1.2) so $M \subset R^{2n-4}$ from [11, Theorem 2.3]. Let v denote the normal bundle to an embedding of M in R^{2n-4}. Note that $\pi_{n-1}(S^{n-5})$ is the 4-stem and so is trivial. Thus the only obstruction to a cross-section of the sphere bundle associated to v is the Euler class $\chi(v)$. But $\chi(v)=0$ since v is the normal bundle to an embedding. So $M \subseteq R^{2n-5}$ by Hirsch [12].

REMARK. Set $m=2^r+1$ for $r>1$. Quaternionic projective space $QP^m \not\subset R^{8m-5}$ from [22] and $QP^m \not\subseteq R^{8m-6}$ from [7] so (1.3) is a best possible result.

PROPOSITION 1.4. *Let K be a complex with dimension $n \equiv 6 \bmod 8$ and $n>14$. Suppose that $H^{n-i}(K)=0$ for $1 \leqq i \leqq 4$. Let ξ be any stable orientable bundle over K. Then* g.dim $\xi \leqq n-7$ *iff* $w_{n-6}(\xi)=0$.

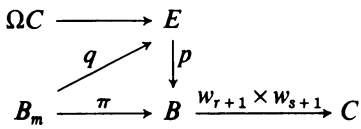
Proof. Write $n=8t+6$ for $t>1$ and let the map $\xi\colon K \to BSO$ classify the bundle ξ. The homotopy groups of the fiber V for the fibration $\pi\colon BSO(8t-1) \to BSO$ are listed in [13]. In particular, $\pi_{8t-1}(V)=Z_2$ while $\pi_{8t}(V)=\pi_{8t+5}(V)=0$. Thus ξ lifts to $BSO(8t-1)$ iff $w_{8t}(\xi)=0$.

THEOREM 1.5. *Let M^n be a 4-connected manifold with $n \equiv 6 \bmod 8$ and $n>14$. Then $M \subseteq R^{2n-7}$ and $M \subset R^{2n-6}$.*

Proof. Let v denote the stable normal bundle of M. By (1.2) $w_{n-6}(v)=0$. Poincaré duality gives $H^{n-i}(M)=0$ for $1 \leqq i \leqq 4$. Thus $M \subseteq R^{2n-7}$ from (1.4) and [12]. Finally $M \subset R^{2n-6}$ by Glover [10].

2. **Cohomology operations and k-invariants.** In [25] Thomas describes a method for expressing k-invariants in a Postnikov resolution for a fibration in terms of higher order cohomology operations applied to classes coming from the base of the fibration. We consider only a Postnikov resolution for the fibration $\pi\colon B_m \to B$ through dimensions $\leqq t$ where π^* is surjective and $m<t<2m$. Here B_m and B denote either $BSO(m)$ and BSO or $B\,\mathrm{Spin}\,(m)$ and $B\,\mathrm{Spin}$ respectively. We derive from the generating class theorem [25, Theorem 5.9] a version for the case of independent second-order k-invariants in a resolution for π. Consider the following commutative diagram.

(2.1)



Here E represents the first stage in a resolution for π through dimensions $\leqq t$ and so p is the principal fiber map classified by the vector (w_{r+1}, w_{s+1}) of Stiefel-Whitney classes. Let ι_r denote the fundamental class of the factor $K(Z_2, r)$ in ΩC and define ι_s similarly. Let $m\colon \Omega C \times E \to E$ denote multiplication in the principal fibration and $\rho\colon \Omega C \times E \to E$ the projection map. Suppose that a class k in $H^p(E)$ for $m < p \leqq t$ is a second-order k-invariant for π independent of w_{s+1}. That is, $\mu(k) = \hat{\alpha} \circ (\iota_r \otimes 1)$ for some class $\hat{\alpha}$ in $A(B)$. The morphism $\mu\colon H^*(E) \to H^*(\Omega C \times E, E)$ is defined in [25] so that $j^* \circ \mu = m^* - \rho^*$ for the inclusion $j\colon \Omega C \times E \to (\Omega C \times E, E)$. From construction of the resolution for π, $\ker p^* = \ker \pi^*$ through dimensions $\leqq t$ so $\ker q^* \cap \ker \mu = 0$ through dimensions $\leqq t$ by [25, Proposition 5.11].

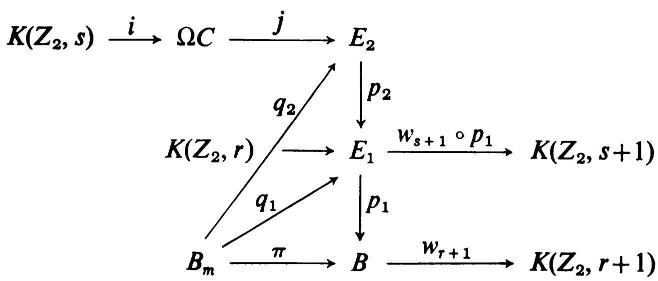
We suppose also that a class v in $H^*(B)$ is a generating class for k. That is, there is a complex K, a map $\eta\colon B \to K$, and a class α in $A(K)$ such that $\hat{\alpha} = \eta^*\alpha$. There are vectors $\beta = (\beta_1, \ldots, \beta_j)$ and $\psi = (\psi_1, \ldots, \psi_j)$ of primary operations over $A(K)$ and a primary operation φ over $A(K)$ such that $(\varphi, \psi)v = (w_{r+1}, 0)$ and there is a relation

$$\alpha \cdot \varphi + \beta \cdot \psi = 0. \tag{2.2}$$

Let Ω be any secondary operation associated to (2.2). Then Ω determines a coset L of $\mathrm{Indet}^p\,(B; \Omega, \eta)$ such that $\Omega(\pi^* v, \pi^*\eta) = \pi^* L$. Under the above assumptions the generating class theorem states

THEOREM 2.3. $k \in \Omega(p^*v, p^*\eta) - p^*L$.

Proof. Consider the following commutative diagram of complexes and maps.



Let $\mu_1\colon H^*(E_1) \to H^*(K(Z_2, r) \times E_1, E_1)$ and $\mu_2\colon H^*(E_2) \to H^*(K(Z_2, s) \times E_2, E_2)$ be the morphisms corresponding to μ for the principal fiber maps p_1 and p_2 respectively. Identify E_2 with E in (2.1) as fiber spaces over B via a fiber-preserving homeomorphism and regard the composite map $p_1 \circ p_2$ as p. Let k_1 in $H^p(E_1)$ be a class such that $\mu_1(k_1) = \hat{\alpha}(\iota \otimes 1)$ where ι is the fundamental class of $K(Z_2, r)$. Note that $\mu(p_2^* k_1) = \hat{\alpha}(\iota_r \otimes 1)$ so $p_2^* k_1 = k$. (See [25] or [30].) There is a class h in $\ker q_1^* \cap \ker \mu_1 \cap H^p(E_1)$ such that $k_1 + h \in \Omega(p_1^* v, p_1^* \eta) - p_1^* L$ by [25, Theorem

5.9]. Since $p_2^*h \in \ker q^* \cap \ker \mu \cap H^p(E) = 0$, the result follows by naturality of Ω.

REMARK 1. Let $\xi: X \to B$ classify a stable vector bundle ξ over a complex X for which $w_{r+1}(\xi) = w_{s+1}(\xi) = 0$. Then ξ lifts to E in (2.1) and, by definition, $k(\xi) = \bigcup_g g^*k$ where g ranges over all liftings of ξ. Note that $k(\xi)$ is a coset of the subgroup $(\hat{\alpha}H^*(X)) \cap H^p(X)$, the indeterminacy subgroup of $k(\xi)$. If $0 \in \Omega(\pi^*v, \pi^*\eta)$ and $\mathrm{Indet}^p(X; \Omega, \xi^*\eta) =$ indeterminacy subgroup of $k(\xi)$, then $k(\xi) = \Omega(\xi^*v, \xi^*\eta)$ from (2.3).

REMARK 2. In applications of (2.3) in §3 and §4 the class v is the Stiefel-Whitney class w_p for p an even integer. Suppose an operation Ω associated to relation (2.2) can be chosen 1-trivial if $B = BSO$ or spin trivial if $B = B\,\mathrm{Spin}$. (See [26].) Let $i: B_p \subset B_m$ denote the standard inclusion. It follows that $\Omega(\pi^*v, \pi^*\eta) \cap \ker i^* \neq \varnothing$ in $H^p(B_m)$ for this choice of Ω from [26, Theorem 3.3].

Versions of Thomas' generating class theorem for expressing a k-invariant lifted to the Thom complex of a bundle by means of an Adams-Maunder operation applied to the Thom class are given in [27], [28], and [21]. An application in §4 uses a stable tertiary operation which we define here. Consider the following stable integral relations and associated secondary operations for $t > 1$.

(2.4) $\quad \Omega_1 : Sq^2Sq^{4t+2} = 0, \qquad \Omega_2 : (Sq^2Sq^1)Sq^{4t+2} + Sq^1Sq^{4t+4} = 0.$

Let Ω denote the 2-valued secondary operation (Ω_1, Ω_2). By [16] Ω_1 and Ω_2 can be chosen so that $\iota \cdot Sq^2\iota \in \Omega_1(\iota)$ and $0 \in \Omega_2(\iota)$ where ι denotes the fundamental class of $K(Z, 4t+1)$. For this choice of Ω [17, Lemma 3.1] states that the relation $Sq^2\Omega_1 + Sq^1\Omega_2 = Sq^{4t+3}Sq^2$ holds stably and with zero indeterminacy between the component operations Ω_i of Ω.

DEFINITION 2.5. A spin integral cohomology class x is an integral cohomology class for which $Sq^2x = 0$.

The fiber E_n of the map

$$K(Z, n) \xrightarrow{Sq^2\iota_n} K(Z_2, n+2)$$

is a classifying space for spin integral classes of dimension n. We regard E_n as $\Omega^m E_{n+m}$ and $e_n = \sigma^m(e_{n+m})$ where e_j is the fundamental class of E_j and σ is the suspension homomorphism.

DEFINITION 2.6. A class z in $H^*(E_n)$ is called stable if, for every positive integer m, there is a class y in $H^*(E_{n+m})$ such that $\sigma^m(y) = z$.

Set $E = E_{4t+1}$ with fundamental class e. Note that $(0, 0) \in \Omega(e)$ for Ω as chosen above. Consider the following stable relation on spin integral classes:

(2.7) $$Sq^2\Omega_1 + Sq^1\Omega_2 = 0.$$

THEOREM 2.8. *A stable tertiary operation* ψ *associated to relation* (2.7) *can be chosen so that* $\lambda e \cdot y \in \psi(e)$ *where* y *generates* $H^{4t+4}(E)$ *and* λ *is in* Z_2.

Proof. The universal example for a tertiary operation associated to (2.7) is a fiber space over E. Thus any choice for ψ can be altered by stable classes in $H^{8t+5}(E)$.

It has a vector space basis over Z_2 consisting of stable classes $Sq^I e$ for certain admissible monomials Sq^I in A and also the nonstable class $e \cdot y$ where y generates $H^{4t+4}(E)$. This follows from the Serre spectral sequence applied to the fibration $r: E \to K(Z, 4t+1)$ with fiber $K(Z_2, 4t+2)$ and classifying map $Sq^2\iota$. The result follows.

Remark. It follows that $\lambda = 1$ by a result of L. Kristensen. An immediate consequence of (2.8) is the following.

Corollary 2.9. *Let X be a complex such that $H^{4t+4}(X) = 0$. A stable tertiary operation ψ associated to relation* (2.7) *can be chosen, independently of X, so that $0 \in \psi(x)$ for every spin integral class x in $H^s(X)$ for $s \leqq 4t+1$.*

In §4 it is necessary to evaluate a stable secondary operation on 1-dimensional classes. Recall that the excess of a homogeneous element θ in the Steenrod algebra A, written ex (θ), is the minimum value of the excesses of the admissible monomials in A whose sum is θ. Consider the following relation in A of degree n:

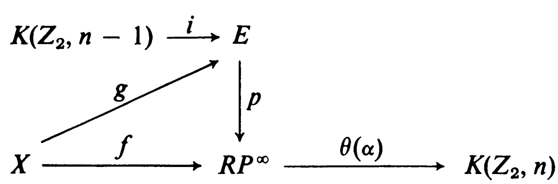
$$Sq^1\theta + \sum_{i=1}^{s} \gamma_i \theta_i = 0 \tag{2.10}$$

where ex $(\theta_i) > 1$ and degree $(\gamma_i) > 1$ for $1 \leqq i \leqq s$. Let Ω be any stable secondary operation associated to (2.10) and let ρ denote mod 2 reduction of integral classes.

Proposition 2.11. *Let X be a complex and x a class in $H^1(X)$ in the domain of Ω such that $x^n = 0$. If* ex $(\theta) > 1$, $0 \in \Omega(x)$. *If* ex $(\theta) = 1$, *then $n = 2^r$ for some integer r and $\rho(u) \in \Omega(x)$ where $2u = y^{2^r - 1}$ in $H^n(X; Z)$ and $\rho(y) = x^2$.*

Proof. Let $f: X \to RP^\infty$ classify x and let α denote the generator of $H^*(RP^\infty)$. If ex $(\theta) > 1$, the functional cohomology operation associated to (2.10) vanishes on α by [2, Teorema 6.6]. It follows from the Peterson-Stein formula [2, Teorema 5.2] and the assumption $x^n = 0$ that $0 \in \Omega(x)$. If ex $(\theta) = 1$, clearly $n = 2^r$ for some integer r and $\theta(\alpha) = \alpha^{2^r}$. Consider the following commutative diagram.

(2.12)



Here p is the principal fiber map classified by the map $\theta(\alpha)$. Since $\alpha^2 = \rho(\beta)$ for β in $H^2(RP^\infty; Z)$, $p^*\beta^{2^r - 1} = 2z$ for z in $H^n(E; Z)$. Further, $i^*\rho(z) = Sq^1\iota$ since this is true in the universal example for division by 2. (See [9].) Applying the Serre spectral sequence to the fiber map p shows that $H^n(E)$ is generated by $\rho(z)$. Set $y = f^*\beta$ and $2u = y^{2^r - 1}$. The universal example for Ω on 1-dimensional classes is a fiber space over RP^∞ fiber homotopically equivalent to $E \times \prod_{i=1}^{s} K(Z_2, \text{degree } \theta_i)$. It follows that $\rho(u) = g^*\rho(z) \in \Omega(x)$.

COROLLARY 2.13. *Let Ω be a stable secondary operation associated to relation* (2.10) *with n even. Let M^n be an orientable manifold. Then any class u in $H^1(M)$ lies in the domain of* Ω *and* $0 \in \Omega(u)$.

Proof. Since M is orientable, $u^n = Sq^1 u^{n-1} = 0$. The result follows from (2.11) and the fact $H^n(M; Z) = Z$.

3. Immersions of k-connected manifolds.

In this section we derive some immersion results for certain k-connected manifolds for small values of k.

PROPOSITION 3.1. *Let K be a complex of dimension $n \equiv 6 \bmod 8$ with $n > 6$. Assume that $H^{n-1}(K) = H^{n-2}(K) = 0$ and $Sq^1 H^{n-4}(K) \subseteq Sq^2 H^{n-5}(K)$. Let ξ be a stable spin bundle over K with $w_{n-6}(\xi) = 0$. Then* g.dim $\xi \leqq n-5$.

Proof. Set $n = 8t+6$ for $t > 0$ and refer to Postnikov resolution III in §5. Now $w_{8t+2}(\xi) = w_{8t+4}(\xi) = 0$ from the Wu relations since $w_{8t}(\xi) = 0$. Let the map $\xi \colon K \to B\,\mathrm{Spin}$ classify the bundle ξ. Thus $k_1^1(\xi)$ is defined and ξ clearly lifts to $B\,\mathrm{Spin}\,(8t+1)$ iff $0 \in k_1^1(\xi)$. Note that k_1^1 is independent of w_{8t+4}. One checks that w_{8t} in $H^*(B\,\mathrm{Spin})$ is a generating class for k_1^1 with respect to the relation

$$Sq^2 Sq^2 + Sq^1(Sq^2 Sq^1) = 0. \tag{3.2}$$

Any secondary operation Ω associated to (3.2) is spin trivial since $B\,\mathrm{Spin}$ is 3-connected. By [26] for any choice of Ω, $0 \in \Omega(w_{8t}) \subseteq H^{8t+3}(B\,\mathrm{Spin}\,(8t+1))$ since $\ker i^* \cap H^{8t+3}(B\,\mathrm{Spin}\,(8t+1)) = 0$ where $i : B\,\mathrm{Spin}\,(8t) \subset B\,\mathrm{Spin}\,(8t+1)$. Set $L = \mathrm{Indet}^{8t+3}\,(B\,\mathrm{Spin}; \Omega)$. Then $k_1^1 \in \Omega(p_1^* w_{8t})$ by Theorem 2.3. The indeterminacy of $k_1^1(\xi) = \mathrm{Indet}^{8t+3}\,(K; \Omega)$ since by hypothesis $Sq^1 H^{8t+2}(K) \subseteq Sq^2 H^{8t+1}(K)$. Thus $0 \in \Omega(w_{8t}(\xi)) = k_1^1(\xi)$.

THEOREM 3.3. *Let M^n be a 2-connected manifold with $n \equiv 6 \bmod 8$ and $n > 6$. Assume $H_3(M; Z)$ has no 2-torsion. Then $M \subseteq R^{2n-5}$ and $M \subset R^{2n-4}$.*

Proof. Let v denote the stable normal bundle of M. By (1.2) $w_{n-6}(v) = 0$. $H^{n-2}(M) = H^{n-1}(M) = 0$ and $H^{n-3}(M; Z)$ has no 2-torsion by Poincaré duality so $Sq^1 H^{n-4}(M) = 0$. Thus g.dim $v \leqq n-5$ by (3.1) and so $M \subseteq R^{2n-5}$ by Hirsch [12]. $M \subset R^{2n-4}$ by Glover [10].

THEOREM 3.4. *Let M^n be a 3-connected manifold with $n \equiv 7 \bmod 8$ and $n > 7$. Suppose $Sq^1 H^{n-5}(M) \subseteq Sq^2 H^{n-6}(M)$. Then $M \subseteq R^{2n-6}$.*

Proof. Write $n = 8t+7$ for $t > 0$ and refer to resolution III in §5. Let $v \colon M \to B\,\mathrm{Spin}$ classify the stable normal bundle of M. By (1.2) $w_{8t}(v) = 0$ so $w_{8t+2}(v) = 0$ from the Wu relations. Clearly v lifts to $B\,\mathrm{Spin}\,(8t+1)$ iff $0 \in k_1^1(v)$ and $k_4^1(v) = 0$. The proof of (3.1) shows that $0 \in k_1^1(v)$.

Note that k_4^1 is independent of w_{8t+2}. Let U_v and T_v denote the Thom class and

Thom complex of v respectively. By [15] we can choose a stable secondary operation Γ associated to the relation

$$Sq^4Sq^{8t+4}+Sq^{8t+7}Sq^1+Sq^{8t+6}Sq^2 = 0$$

such that $u\cdot Sq^4u \in \Gamma(u)$ for any class u of dimension $8t+3$ in the domain of Γ. Applying the technique for isolating an independent k-invariant from a resolution in [28] and the generating class theorem [28, Theorem 6.5] gives the result $U_v\cdot k_4^1(v)=\Gamma(U_v)$ in $H^*(T_v)$. But the top class in $H^*(T_v)$ is spherical by [16] so $\Gamma(U_v)=0$. Thus g.dim $v\leqq 8t+1$ and the result follows by [12].

Let $M^n=S^3\times CP^{2^r+1}$ for $r>1$. It follows from [7] that $M\nsubseteq R^{2n-7}$ so the following result is best possible.

THEOREM 3.5. *Let M^n be a simply connected spin manifold with $n\equiv 5 \bmod 8$ and $n>13$. Suppose the following conditions hold*:

1. $x^2=0$ *iff* $x=0$ *for any x in $H^2(M)$.*
2. $y^2=0$ *iff* $Sq^2y=0$ *for any y in $H^3(M)$.*
3. $\bar{w}_{n-6}(M)=0$ *if* $n=2^r+5$.

Then M immerses in R^{2n-6}.

Proof. Write $n=8t+5$ for $t>1$ and refer to Postnikov resolution IV in §5. Let $v\colon M\to B$ Spin classify the stable normal bundle of M. Note that $w_{8t}(v)=0$ by (1.2) so v lifts to E_1. A simple argument using Poincaré duality and the Wu classes shows that $Sq^2\colon H^{n-4}(M)\to H^{n-2}(M)$ is an epimorphism iff condition 1 holds. If v lifts to E_2, $0\in k_1^2(v)$ since the indeterminacy subgroup of $k_1^2(v)=Sq^2H^{n-4}(M)$. Since M is simply connected, v lifts to B Spin $(8t-1)$ iff $0\in k_1^1(v)$ and $k_3^1(v)=0$.

The functional cohomology operation associated to the relation

$$(Sq^4Sq^2)Sq^{8t}+Sq^{8t+4}Sq^2+Sq^{8t+3}(Sq^2Sq^1) = 0 \tag{3.6}$$

vanishes on classes of dimension $<8t$ in its domain by [2, Teorema 6.6]. By the Peterson-Stein formula [2, Teorema 5.2] a stable secondary operation Γ associated to (3.6) can be chosen independently of u so that $\lambda u\cdot Sq^6u\in\Gamma(u)$ for fixed λ in Z_2 where u is any class of dimension $8t-1$ in the domain of Γ. Applying the generating class theorem [28, Theorem 6.5] gives

$$U_{E_1}\cdot(k_3^1+\lambda p_1^*(w_6w_{8t-1}))\in\Gamma(U_{E_1}).$$

Note that $Sq^2(w_4\cdot w_{8t-1})=w_6w_{8t-1}+w_4w_{8t+1}$ so $w_6(v)w_{8t-1}(v)=0$. Thus $U_v\cdot k_3^1(v)=\Gamma(U_v)$ since $\Gamma(U_v)$ has zero indeterminacy. But the top class in $H^*(T_v)$ is spherical by [16] so $k_3^1(v)=0$.

One checks that w_{8t-2} in $H^*(B$ Spin) is a generating class for k_1^1 with respect to the relation

$$Sq^2(Sq^2Sq^1) = 0. \tag{3.7}$$

Let Ω be the spin trivial stable secondary operation associated to (3.7). (See [26].) By [26, Theorem 3.3] (or Remark 2 in §2) $0\in\Omega(w_{8t-2})$ in $H^*(B$ Spin $(8t-1))$ since

$\ker i^* \cap H^{8t+2}(B\,\mathrm{Spin}\,(8t-1))=0$ where $i: B\,\mathrm{Spin}\,(8t-2)\subset B\,\mathrm{Spin}\,(8t-1)$. Thus $k_1^1 \in \Omega(p_1^* w_{8t-2})$ by the generating class theorem [25, Theorem 5.9]. It follows from Poincaré duality and the Wu classes that condition 2 holds iff $Sq^2H^{8t}(M) = Sq^2Sq^1H^{8t-1}(M)$. So $k_1^1(v)=\Omega(w_{8t-2}(v))$. But from [26] $\Omega=\varphi\circ\delta$ where φ is the unique secondary operation associated to the integral relation $Sq^2Sq^2=0$ and δ is the Bockstein operator. Since $\bar{w}_{n-6}(M)=0$ from [18] and condition 3, it follows that $0\in\varphi(\bar{w}_{n-6}(M))=\varphi(\delta w_{8t-2}(v))=\Omega(w_{8t-2}(v))=k_1^1(v)$. So g.dim $v\leqq n-6$ and the result follows by Hirsch [12].

4. **Orientable and spin manifolds.** In this section we establish immersions for some orientable and spin manifolds. QP^n has a best possible immersion in R^{8n-3} for $n=2^r$ by [16]. CP^m does not immerse in R^{4m-3} for $m=2^r+2^s$ with $r>s>0$ by [22]. For spin manifolds we prove the following

THEOREM 4.1. *Let M^n be a spin manifold with $n\equiv 0$ mod 4. Then M immerses in R^{2n-3} for n not a power of 2. M immerses in R^{2n-3} iff $\bar{w}_{n-2}(M)=0$ for $n=2^r$ with $r>3$.*

Proof. Set $n=4t+4$ for $t>1$ and refer to Postnikov resolution I in §5. Let $v: M\to B\,\mathrm{Spin}$ classify the stable normal bundle v of M. Now $w_{4t+2}(v)=w_{4t+4}(v) =0$ by (1.1) and the assumption $\bar{w}_{n-2}(M)=0$ for $n=2^r$. Note that $k_1^1(v)$ and $k_2^1(v)$ have zero indeterminacy. Let U_{4t+1} denote the Thom class associated to the universal bundle γ_{4t+1} over $B\,\mathrm{Spin}\,(4t+1)$. Let $\Omega=(\Omega_1,\Omega_2)$ be the double secondary operation associated to relation (2.4) such that $(0,0)\in\Omega(e)$. (See §2.) Thus $(0,0)\in\Omega(U_{4t+1})$. Let T_v and U_v denote the Thom complex and Thom class associated to v respectively. Applying a version of the generating class theorem [27, Theorem 6.4] for expressing simultaneously two second-order k-invariants lifted to the Thom complex and then checking indeterminacies gives the result that $U_v\cdot(k_1^1(v), k_2^1(v))=\Omega(U_v)$. (See also [21].) But $U_v\cdot k_2^1(v)=\Omega_2(U_v)=0$ since the top class in $H^*(T_v)$ is spherical by [16]. We apply a duality theorem of Adem-Gitler [3, Theorem 5.1] in order to show $k_1^1(v)=0$. Let Γ denote the secondary operation dual to Ω_1 and associated to the relation

$$c(Sq^{4t+2})Sq^2+Sq^1c(Sq^{4t+3}) = 0 \tag{4.2}$$

where c is the anti-automorphism of A. Then $\Omega_1(U_v)=0$ iff Γ vanishes on its domain of definition in $H^1(M)$ from [3]. By (2.13) Γ vanishes on every class in $H^1(M)$ so $k_1^1(v)=0$.

The k-invariant k_1^2 can be expressed by the tertiary operation ψ of Theorem 2.8. Since $B\,\mathrm{Spin}\,(4t+1)$ is 3-connected, $0\in\psi(U_{4t+1})$ by (2.9). Note that $k_1^2(v)$ has zero indeterminacy. Applying a version of the generating class theorem for a third-order k-invariant lifted to the Thom complex [21, Proposition 4.6] gives the result $U_v\cdot k_1^2(v)\in\psi(U_v)$. But $\psi(U_v)=0$ since the top class in $H^*(T_v)$ is spherical by [16] so $k_1^2(v)=0$. Thus v lifts to $B\,\mathrm{Spin}\,(4t+1)$ and the result follows by Hirsch [12].

REMARK. It follows from [31, Lemma 1] that a 4-dimensional spin manifold immerses in R^5.

Thomas proves in [26] that a spin manifold M^n immerses in R^{2n-4} for $n\equiv 3 \bmod 8$ and $n>3$.

THEOREM 4.3. *Let M^n be a spin manifold with $n\equiv 7 \bmod 8$ and $n>7$. Let ξ be a stable spin bundle over M. If $w_{n-7}(\xi)=0$, g.dim $\xi\leqq n-4$. Thus $M\subseteq R^{2n-4}$.*

Proof. Set $n=8t+7$ and refer to resolution II in §5. Let $\xi: M\to B\,\mathrm{Spin}$ classify the bundle ξ. Now $w_{8t+4}(\xi)=0$ since $w_{8t+4}=Sq^4w_{8t}+w_4\cdot w_{8t}$ in $H^*(B\,\mathrm{Spin})$. We express k_2^1 by means of a twisted secondary operation due to Thomas. Consider the following relation in $A(K(Z_2, 4))$:

$$\gamma\cdot\gamma+Sq^2(\gamma\cdot Sq^2)+Sq^1(Sq^2\gamma Sq^1)+\delta\cdot(Sq^2Sq^1) = 0 \tag{4.4}$$

where $\gamma=\iota\otimes 1+1\otimes Sq^4$ and $\delta=Sq^1\iota\otimes 1$. Let $w_4: B\,\mathrm{Spin}\to K(Z_2, 4)$ classify the Stiefel-Whitney class w_4. One checks that w_{8t} in $H^*(B\,\mathrm{Spin})$ is a generating class for k_2^1 with respect to the relation (4.4). Let φ be a secondary operation associated to (4.4). Let U_s denote the Thom class associated to the universal bundle γ_s over $B\,\mathrm{Spin}\,(s)$ for $s>7$. Clearly $\varphi(U_s, w_4)$ is defined and φ is spin trivial since $U_s\cdot w_7=Sq^1(U_s\cdot w_6)$. Let $j: B\,\mathrm{Spin}\,(8t)\subset B\,\mathrm{Spin}\,(8t+3)$ denote the standard inclusion. Since $\ker j^*\cap H^{8t+7}(B\,\mathrm{Spin}\,(8t+3))$ is generated by w_6w_{8t+1} and w_4w_{8t+3}, it follows from Remark 2 in §2 that

$$\lambda_1w_6w_{8t+1}+\lambda_2w_4w_{8t+3}\in\varphi(w_{8t}, w_4)$$

in $H^{8t+7}(B\,\mathrm{Spin}\,(8t+3))$ for some λ_1 and λ_2 in Z_2. Since $Sq^5Sq^2w_{8t}=w_4w_{8t+3}$ and $Sq^2(w_4\cdot Sq^1w_{8t})=w_6w_{8t+1}$, one has $0\in\Gamma(w_{8t}, w_4)$ in $H^*(B\,\mathrm{Spin}\,(8t+3))$ for

$$\Gamma = \varphi+\lambda_1(1\otimes Sq^5Sq^2)+\lambda_2Sq^2(\iota\otimes Sq^1).$$

The generating class theorem [25, Theorem 5.9] gives $k_2^1\in\Gamma(p_1^*w_{8t}, p_1^*w_4)$. Now $\mathrm{Indet}^{8t+7}(M;\Gamma, w_4(\xi))=(Sq^4+\cdot w_4(\xi))H^{n-4}(M)=$ indeterminacy of $k_2^1(\xi)$. Thus $0\in\Gamma(w_{8t}(\xi), w_4(\xi))=k_2^1(\xi)$. The proof of Theorem 1.3 in [26] shows that $k_1^1(\xi)=k_1^2(\xi)=0$. Thus ξ lifts to $B\,\mathrm{Spin}\,(8t+3)$. By (1.2) $\bar{w}_{n-7}(M)=0$ so g.dim $v\leqq n-4$ where v denotes the stable normal bundle of M.

Theorem 4.3 has an immediate application to a problem investigated by Thomas in [29]. Here we require a manifold M to mean only a smooth connected manifold without boundary. Let $\tau_0(M)$ and $v(M)$ denote the stable tangent and normal bundles of a manifold M respectively. Given a map $f: M\to N$ between manifolds, define the stable bundle $v_f=f^*\tau_0(N)+v(M)$. The map f is called a spin map if $f^*w_1(N)=f^*w_2(N)=0$ and M is a closed spin manifold.

THEOREM 4.5. *Let M^{8t+7} and N^{16t+10} be manifolds with $t>0$. Suppose $f: M\to N$ is a spin map. If $w_{8t}(v_f)=0$, then f is homotopic to an immersion.*

Proof. Note that v_f is a stable spin bundle over M. By (4.3) g.dim $v_f \leqq 8t+3 = \dim N - \dim M$. The result follows from the formulation of Hirsch's theorem in [29].

Manifolds again are assumed to be closed. We prove

THEOREM 4.6. *Let M^n be an orientable manifold with $n \equiv 1 \bmod 4$ and $n > 9$. Suppose the following conditions hold:*

1. *$u^2 = 0$ iff $u = 0$ for any u in $H^1(M)$.*
2. *$w_2(M) = u^2$ for some u in $H^1(M)$ iff $w_2(M) = 0$.*
3. *$Sq^1 y = 0$ for any y in $H^2(M)$ such that $y^2 = 0$.*
4. *$\bar{w}_{n-5}(M) = 0$ if $\alpha(n) < 5$.*

Then M immerses in R^{2n-4}.

Proof. Write $n = 4t+5$ and refer to Postnikov resolution V in §5. Let $v: M \to B$ classify the stable normal bundle v of M.

Case I. $B = BSO$ and $w_2(M) \neq 0$.

Condition 4 and [18] give $w_{4t+2}(v) = w_{4t+4}(v) = 0$. Condition 1 is equivalent to $Sq^1 H^{4t+3}(M) = H^{4t+4}(M)$ from Poincaré duality and the Wu relations. So $0 \in k_2^1(v)$ since $Sq^1 H^{4t+3}(M) =$ indeterminacy of $k_2^1(v)$. Note that $0 \in k_1^2(v)$ also through indeterminacy if v lifts to E_2. Let $g: M \to E_1$ be a lifting of v such that $g^* k_2^1 = 0$. Condition 2 is equivalent to the condition $Sq^2 y \neq 0$ and $Sq^1 y = 0$ for some class y in $H^{n-2}(M)$. Alter g, if necessary, to give a lifting $h: M \to E_1$ of v such that $h^* k_3^1 = 0$ and $h^* k_2^1 = g^* k_2^1 = 0$. Note that

$$(Sq^2 + \cdot w_2(M)) Sq^1 H^{n-4}(M) = 0 = (Sq^4 + \cdot \bar{w}_4(M)) H^{n-4}(M).$$

Thus v lifts to $BSO(4t+1)$ iff $0 \in k_1^1(v)$. Assume that $\alpha(n) > 4$. Let φ be the secondary operation associated to the relation—$Sq^2 Sq^{4t+2} + Sq^{4t+3} Sq^1 = 0$—such that $u \cdot Sq^2 u \in \varphi(u)$ for any class u of dimension $4t+1$ in the domain of φ. The generating class theorem [28, Theorem 6.5] gives the result $U_{E_1} \cdot (k_1^1 + p_1^* w_2 w_{4t+1}) \in \varphi(U_{E_1})$. But $w_{4t+1}(v) = 0$ for $\alpha(n) > 4$ by [18] so $U_v \cdot k_1^1(v) = \varphi(U_v)$. Let Γ denote the operation dual to φ associated to the relation (4.2). By [3] φ vanishes on U_v iff Γ vanishes on its domain of definition in $H^2(M)$. Recall from [18] that a homogeneous element θ of degree $r-s$ in the Steenrod algebra A vanishes on s-dimensional classes if $\alpha(r) > s$. Thus the functional cohomology operation ψ associated to relation (4.2) vanishes on 2-dimensional classes since $\alpha(4t+5) > 4$. (See [2].) It follows from condition 3 and the Peterson-Stein formula in [2] that $\Gamma(u) = \psi(u)$ for any class u in $H^2(M)$ in the domain of Γ. So $0 \in k_1^1(v)$ for $\alpha(n) > 4$.

Suppose now that $\alpha(n) < 5$. Consider the relation in $A(K(Z_2, 2))$:

$$\beta \cdot \beta + Sq^1 \cdot (\beta \cdot Sq^1) = 0 \tag{4.7}$$

where $\beta = \iota \otimes 1 + 1 \otimes Sq^2$. Let the map $w_2: BSO \to K(Z_2, 2)$ induce an $A(K(Z_2, 2))$-module structure on $H^*(BSO)$. By §6 of [25] a twisted secondary operation Ω

associated to (4.7) can be chosen so that $k_1^1 \in \Omega(p_1^* w_{4t}, p_1^* w_2)$. Condition 3 is equivalent to the condition $Sq^1 H^{n-3}(M) \subseteq (Sq^2 + \cdot w_2(M))H^{n-4}(M)$. So

$$0 \in \Omega(w_{4t}(v), w_2(v)) = k_1^1(v)$$

by condition 4. Thus v lifts to $BSO(4t+1)$.

Case II. $B = B\,\mathrm{Spin}$ and $w_2(M)=0$. The only essential difference from Case I is the computation of $k_3^1(v)$. We choose by [15] the secondary operation Γ associated to the stable integral relation

$$Sq^4 Sq^{4t+2} + Sq^{4t+4} Sq^2 + t Sq^2 Sq^{4t+4} = 0 \tag{4.8}$$

such that $u \cdot Sq^4 u \in \Gamma(u)$ for any spin integral class u of dimension $4t+1$. By the generating class theorem $U_{E_1} \cdot (k_3^1 + p_1^* w_4 w_{4t+1}) \in \Gamma(U_{E_1})$. $Sq^1(w_4 w_{4t}) = w_4 w_{4t+1}$ so $w_4(v) \cdot w_{4t+1}(v) = 0$. Thus $0 = \Gamma(U_v) = U_v \cdot k_3^1(v)$ since the top class in $H^*(T_v)$ is spherical by [16]. So $k_3^1(v)=0$ and the result follows.

Refer to [6] and [32] for the cohomology ring and total Stiefel-Whitney class of the Dold manifold $P(m, n)$. A consequence of Theorem 4.6 is the following

COROLLARY 4.9. *Set $N = m+2n$. Let $P(m,n)$ be any orientable Dold manifold with $N \equiv 1 \bmod 4$, $m>1$, $n>0$, and $n \neq 2^r$ when $\alpha(N) \leqq 3$. Then $P(m,n) \subseteq R^{2N-4}$.*

5. **Postnikov resolutions.** These Postnikov resolutions for the fiber map $\pi: B_m \to B$ are constructed by the techniques of [24]. We refer the reader also to [14] and [8] for the theory and construction of modified Postnikov resolutions. The homotopy groups of the fiber for π appear in [13] and [20]. The tower of spaces is displayed only for resolution I. The k-invariant k_j^i represents a class in $H^*(E_i)$ whose defining relation is a relation in $H^*(E_{i-1})$ where $E_0 = B$.

5.1. Postnikov resolution I for the fibration $\pi: B\,\mathrm{Spin}\,(4t+1) \to B\,\mathrm{Spin}$ for stable spin bundles over complexes of dimension $\leqq 4t+4$ for $t>1$.

$$\begin{array}{lll}
B\,\mathrm{Spin}\,(4t+1) & & \\
\big\downarrow q_3 & & \\
E_3 & & \\
\big\downarrow p_3 & & \\
E_2 & \xrightarrow{k_1^2} & K(Z_2, 4t+4) \\
\big\downarrow p_2 & & \\
E_1 & \xrightarrow{k_1^1 \times k_2^1} & K(Z_2, 4t+3) \times K(Z_2, 4t+4) \\
\big\downarrow p_1 & & \\
B\,\mathrm{Spin} & \xrightarrow{w_{4t+2} \times w_{4t+4}} & K(Z_2, 4t+2) \times K(Z_2, 4t+4)
\end{array}$$

Defining relations for k-invariants:

$$k_1^1\colon\ Sq^2 w_{4t+2} = 0,$$
$$k_2^1\colon Sq^2 Sq^1 w_{4t+2} + Sq^1 w_{4t+4} = 0,$$
$$k_1^2\colon\ Sq^2 k_1^1 + Sq^1 k_2^1 = 0.$$

5.2. Postnikov resolution II for the fibration $\pi\colon B\,\text{Spin}\,(8t+3) \to B\,\text{Spin}$ for stable spin bundles over complexes of dimension $\leqq 8t+7$ for $t>0$.

Defining relations for k-invariants:

$$k_1^0 = w_{8t+4},$$
$$k_1^1\colon\ Sq^2 Sq^1 w_{8t+4} = 0,$$
$$k_2^1\colon\ (Sq^4 + \cdot w_4) w_{8t+4} = 0,$$
$$k_1^2\colon\ Sq^2 k_1^1 = 0.$$

5.3. Postnikov resolution III for the fibration $\pi\colon B\,\text{Spin}\,(8t+1) \to B\,\text{Spin}$ for stable spin bundles over complexes of dimension $\leqq 8t+7$ for $t>0$.

Defining relations for k-invariants:

$$k_1^0 = w_{8t+2}, \qquad k_2^0 = w_{8t+4},$$
$$k_1^1\colon\ Sq^2 w_{8t+2} = 0,$$
$$k_2^1\colon\ Sq^2 Sq^1 w_{8t+2} + Sq^1 w_{8t+4} = 0,$$
$$k_3^1\colon\ (Sq^4 + \cdot w_4) w_{8t+2} = 0,$$
$$k_4^1\colon\ (Sq^4 + \cdot w_4) w_{8t+4} = 0,$$
$$k_1^2\colon\ Sq^2 k_1^1 + Sq^1 k_2^1 = 0.$$

5.4. Postnikov resolution IV for the fibration $\pi\colon B\,\text{Spin}\,(8t-1) \to B\,\text{Spin}$ for stable spin bundles over complexes of dimension $\leqq 8t+5$ for $t>1$. Defining relations for k-invariants:

$$k_1^0 = w_{8t},$$
$$k_1^1\colon\ Sq^2 Sq^1 w_{8t} = 0,$$
$$k_2^1\colon\ (Sq^4 + \cdot w_4) Sq^1 w_{8t} = 0,$$
$$k_3^1\colon\ (Sq^4 + \cdot w_4) Sq^2 w_{8t} = 0,$$
$$k_1^2\colon\ Sq^2 k_1^1 = 0,$$
$$k_2^2\colon\ Sq^2 Sq^1 k_1^1 + Sq^1 k_2^1 = 0,$$
$$k_1^3\colon\ Sq^2 k_1^2 + Sq^1 k_2^2 = 0.$$

5.5. Postnikov resolution V for the fibration $\pi\colon B(4t+1) \to B$ for stable orientable and spin bundles over complexes of dimension $\leqq 4t+5$ for $t>1$. Defining relations for k-invariants:

$$B = BSO,$$
$$k_1^0 = w_{4t+2},$$
$$k_2^0 = w_{4t+4},$$
$$k_1^1\colon\ (Sq^2 + \cdot w_2)w_{4t+2} = 0,$$
$$k_2^1\colon\ (Sq^2 + \cdot w_2)Sq^1 w_{4t+2} + Sq^1 w_{4t+4} = 0,$$
$$k_3^1\colon\ (Sq^4 + \cdot w_4)w_{4t+2} + Sq^2 w_{4t+4} = 0, \quad t \text{ odd},$$
$$k_3^1\colon\ (Sq^4 + \cdot w_4)w_{4t+2} + w_2 w_{4t+4} = 0, \quad t \text{ even},$$
$$k_1^2\colon\ (Sq^2 + \cdot w_2)k_1^1 + Sq^1 k_2^1 = 0.$$

The k-invariants for $B=B$ Spin are obtained by deleting w_2 in the above defining relations.

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